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Approximate controllability and regularity for nonlinear differential equations

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Abstract

In this article, we deal with the existence, uniqueness, and a variation of solutions of the nonlinear control system with nonlinear monotone hemicontinuous and coercive operator. Moreover, the approximate controllability for the given nonlinear control system is studied.

Keywords: nonlinear differential equation, regularity, reachable set, degree theory, approximately controllable

1 Introduction

Let H and V be two real separable Hilbert spaces such that V is a dense subspace of H. We are interested in the following nonlinear differential control system on H:

$$\begin{cases} x'(t) + Ax(t) = g(t, x_t, \int_0^t k(t, s, x_s)ds) + (Bu)(t), \quad 0 < t, \\ x(0) = \phi^0, \quad x(s) = \phi^1(s) \quad -h \le s \le 0, \end{cases} \tag{SE}$$

where the nonlinear term, which is a Lipschitz continuous operator, is a semilinear version of the quasi-linear form. The principal operator A is assumed to be a single valued, monotone operator, which is hemicontinuous and coercive from V to V^*. Here, V^* stands for the dual space of V. Let U be a Banach space of control variables. The controller B is a linear-bounded operator from a Banach space $L^2(0, T; U)$ to $L^2(0, T; H)$ for any $T > 0$. Let the nonlinear mapping k be Lipschitz continuous from $\mathbb{R} \times [-h, 0] \times V$ into H. If the right-hand side of the equation (SE) belongs to $L^2(0, T; V^*)$, then it is well known as the quasi-autonomous differential equation(see Theorem 2.6 of Chapter III in [1]).

The problem of existence for solutions of semilinear evolution equations in Banach spaces has been established by several authors [1-3]. We refer to [2,4,5] to see the existence of solutions for a class of nonlinear evolution equations with monotone perturbations

First, we begin with the existence, and a variational constant formula for solutions of the equation (SE) on $L^2(0, T; V) \cap W^{1,2}(0, T; V^*)$, which is also applicable to optimal control problem. We prove the existence and uniqueness for solution of the equation by converting the problem into a fixed point problem. Thereafter, based on the regularity results for solutions of (SE), we intend to establish the approximate controllability for (SE). The controllability results for linear control systems have been proved by many authors, and several authors have extended these concepts to infinite dimensional semilinear system (see [6-8]). In recent years, as for the controllability for semilinear differential equations, Carrasco and Lebia [9] discussed sufficient conditions for approximate controllability of parabolic equations with delay, and Naito [10] and other authors [6-8,11] proved the approximate controllability under the range conditions of the controller B.

The previous results on the approximate controllability of a semilinear control system have been proved as a particular case of sufficient conditions for the approximate solvability of semilinear equations, assuming either that the semigroup generated by A is a compact operator or that the corresponding linear system (SE) when $g \equiv 0$ is approximately controllable. However, Triggian [12] proved that the abstract linear system is never exactly controllable in an infinite dimensional space when the semigroup generated by A is compact. Thus, we will establish the approximate controllability under more general conditions on the nonlinear term and the controller.

Our aim in this article is to establish the approximate controllability for (SE) under a stronger assumption that $\{y : y(t) = (Bu)(t),\ u \in L^2(0, T; U)\}$ is dense subspace of $L^2(0, T, H)$, which is reasonable and widely used in case of the nonlinear system. We show that the input to solution (control to state) map is compact by using the fact that $L^2(0, T; V) \cap W^{1,2}(0, T; V^*)$ furnished with the usual topology is compactly embedded in $L^2(0, T, H)$, provided that the injection $V \subset H$ is compact.

In the last section, we give a simple example to which the range conditions of the controller can be applied.

2 Nonlinear functional equations

Let H and V be two real Hilbert spaces. Assume and V is dense subspace in H and the injection of V into H is continuous. If H is identified with its dual space, then we may write $V \subset H \subset V^*$ densely, and the corresponding injections are continuous. The norm on V (resp. H) will be denoted by $||\cdot||$ (resp. $|\cdot|$). The duality pairing between the element v_1 of V^* and the element v_2 of V is denoted by (v_1, v_2), which is the ordinary inner product in H if $v_1, v_2 \in H$. For the sake of simplicity, we may consider

$$||u||_* \leq |u| \leq ||u||, \quad u \in V$$

where $||\cdot||_*$ is the norm of the element of V^*. If an operator A is bounded linear from V to V^* and generates an analytic semigroup, then it is easily seen that

$$H = \{x \in V^* : \int_0^T ||Ae^{tA}x||_*^2 dt < \infty\},$$

for the time $T > 0$. Therefore, in terms of the intermediate theory we can see that

$$(V, V^*)_{\frac{1}{2},2} = H$$

where $(V, V^*)_{\frac{1}{2},2}$ denotes the real interpolation space between V and V^*.

We note that a nonlinear operator A is said to be hemicontinuous on V if

$$w - \lim_{t \to 0} A(x + ty) = Ax$$

for every $x, y \in V$ where "w - lim" indicates the weak convergence on V^*. Let $A : V \to V^*$ be given a single-valued, monotone operator and hemicontinuous from V to V^* such that

(A1) $A(0) = 0$, $(Au - Av, u - v) \geq \omega_1 ||u - v||^2 - \omega_2 |u - v|^2$,

(A2) $||Au||_* \leq \omega_3(||u|| + 1)$

for every $u, v \in V$ where $\omega_2 \in \mathbb{R}$ and ω_1, ω_3 are some positive constants.

Here, we note that if $0 \neq A(0)$, then we need the following assumption:

$$(Au, u) \geq \omega_1 ||u||^2 - \omega_2 |u|^2$$

for every $u \in V$. It is also known that A is maximal monotone, and $R(A) = V^*$ where $R(A)$ denotes the range of A.

Let the controller B is a bounded linear operator from a Banach space $L^2(0, T; U)$ to $L^2(0, T; H)$ where U is a Banach space.

For each $t \in [0, T]$, we define $x_t : [-h, 0] \to H$ as

$$x_t(s) = x(t + s), \quad -h \leq s \leq 0.$$

We will set

$$\Pi = L^2(-h, 0; V) \quad \text{and} \quad \mathbb{R}^+ = [0, \infty).$$

Let $\mathcal{L}$ and $\mathcal{B}$ be the Lebesgue σ-field on $[0, \infty)$ and the Borel σ-field on $[-h, 0]$ respectively. Let $k : \mathbb{R}^+ \times \mathbb{R}^+ \times \Pi \to H$ be a nonlinear mapping satisfying the following:

(K1) For any $x. \in \Pi$ the mapping $k(\cdot, \cdot, x.)$ is strongly $\mathcal{L} \times \mathcal{B}$ -measurable;

(K2) There exist positive constants K_0, and K_1 such that

$$|k(t, s, x.) - k(t, s, y.)| \leq K_1 ||x. - y.||_\Pi,$$
$$|k(t, s, 0)| \leq K_0$$

for all $(t, s) \in \mathbb{R}^+ \times [-h, 0]$ and $x., y. \in \Pi$.

Let $g : \mathbb{R}^+ \times \Pi \times H \to H$ be a nonlinear mapping satisfying the following:

(G1) For any $x \in \Pi$, $y \in H$ the mapping $g(\cdot, x., y)$ is strongly $\mathcal{L}$ -measurable;

(G2) There exist positive constants L_0, L_1, and L_2 such that

$$|g(t, x., y) - g(t, \hat{x}., \hat{y})| \leq L_1 ||x. - \hat{x}.||_\Pi + L_2 |y - \hat{y}|,$$
$$|g(t, 0, 0)| \leq L_0$$

for all $t \in \mathbb{R}^+$, $x, \hat{x}. \in \Pi$, and $y, \hat{y} \in H$.

Remark 2.1. *The above operator g is the semilinear case of the nonlinear part of quasi-linear equations considered by Yong and Pan* [13].

For $x \in L^2(-h, T; V)$, $T > 0$ *we set*

$$G(t, x) = g(t, x_t, \int_0^t k(t, s, x_s) ds).$$

Here, as in [13], *we consider the Borel measurable corrections of* $x(\cdot)$.

Lemma 2.1. *Let* $x \in L^2(-h, T; V)$. *Then, the mapping* $t \mapsto x_t$ *belongs to* $C([0, T]; \Pi)$ *and*

$$||x.||_{L^2(0.T;\Pi)} \leq \sqrt{T} ||x||_{L^2(-h,T;V)}. \tag{2.1}$$

Proof. It is easy to verify the first paragraph and (2.1) is a consequence of the estimate:

$$\begin{aligned} ||x.||^2_{L^2(0.T;\Pi)} &\leq \int_0^T ||x_t||^2_\Pi dt \leq \int_0^T \int_{-h}^0 ||x(t + s)||^2 ds dt \\ &\leq \int_0^T dt \int_{-h}^T ||x(s)||^2 ds \leq T ||x||^2_{L^2(-h,T;V)}. \end{aligned}$$

Lemma 2.2. *Let $x \in L^2(-h, T; V)$, $T > 0$. Then, $G(\cdot, x) \in L^2(0, T; H)$ and*

$$\begin{aligned}&||G(\cdot,x)||_{L^2(0,T;H)} \le L_0\sqrt{T} + L_2K_0T^{3/2}/\sqrt{3}\\ &+(L_1\sqrt{T} + L_2K_1T^{3/2}/\sqrt{2})||x||_{L^2(-h,T;V)}.\end{aligned} \tag{2.2}$$

Moreover, if $x_1, x_2 \in L^2(-h, T; V)$, then

$$||G(\cdot,x_1) - G(\cdot,x_2)||_{L^2(0,T;H)} \le (L_1\sqrt{T} + L_2K_1T^{3/2}/\sqrt{2})||x_1 - x_2||_{L^2(-h,T;V)}. \tag{2.3}$$

Proof. It follows from (K2) and (2.1) that

$$\begin{aligned}&||\int_0^{\cdot} k(\cdot,s,x_s)ds||_{L^2(0,T;H)} \le ||\int_0^{\cdot} k(\cdot,s,0)ds||_{L^2(0,T;H)}\\ &+||\int_0^{\cdot} (k(\cdot,s,x_s) - k(\cdot,s,0))ds||_{L^2(0,T;H)}\\ &\le K_0T^{3/2}/\sqrt{3} + \{\int_0^T |\int_0^t K_1||x_s||_\Pi ds|^2 dt\}^{1/2}\\ &\le K_0T^{3/2}/\sqrt{3} + \{\int_0^T K_1^2 t\int_0^t ||x_s||_\Pi^2 ds dt\}^{1/2}\\ &\le K_0T^{3/2}/\sqrt{3} + K_1T/\sqrt{2}||x.||_{L^2(0,T;\Pi)}\\ &\le K_0T^{3/2}/\sqrt{3} + K_1T^{3/2}/\sqrt{2}||x||_{L^2(-h,T;V)}\end{aligned}$$

and hence, from (G2), (2.1), and the above inequality, it is easily seen that

$$\begin{aligned}&||G(\cdot,x)||_{L^2(0,T;H)} \le ||G(\cdot,0)|| + ||G(\cdot,x) - G(\cdot,0)||\\ &\le L_0\sqrt{T} + L_1||x.||_{L^2(0,T;\Pi)} + L_2||\int_0^{\cdot} k(\cdot,s,x_s)ds||_{L^2(0,T;H)}\\ &\le L_0\sqrt{T} + L_1\sqrt{T}||x||_{L^2(-h,T;V)}\\ &+L_2(K_0T^{3/2}/\sqrt{3} + K_1T^{3/2}/\sqrt{2}||x||_{L^2(-h,T;V)}).\end{aligned}$$

Similarly, we can prove (2.3).

Let us consider the quasi-autonomous differential equation

$$\begin{cases} x'(t) + Ax(t) = f(t), & 0 < t \le T,\\ x(0) = \phi^0 \end{cases} \tag{E}$$

where A satisfies the hypotheses mentioned above. The following result is from Theorem 2.6 of Chapter III in [1].

Proposition 2.1. *Let $\Phi^0 \in H$ and $f \in L^2(0, T; V^*)$. Then, there exists a unique solution x of (E) belonging to*

$$C([0,T];H) \cap L^2(0,T;V) \cap W^{1,2}(0,T;V^*)$$

and satisfying

$$|x(t)|^2 + \int_0^t ||x(s)||^2 ds \le C_1(|\phi^0|^2 + \int_0^t ||f(s)||_*^2 ds), \tag{2.4}$$

$$\int_0^t ||\frac{dx(s)}{ds}||_*^2 dt \le C_1(|\phi^0|^2 + \int_0^t ||f(s)||_*^2 ds) \tag{2.5}$$

where C_1 is a constant.

Acting on both sides of (E) by $x(t)$, we have

$$\frac{1}{2}\frac{d}{dt}|x(t)|^2 + \omega_1||x(t)||^2 \le \omega_2|x(t)|^2 + (f(t), x(t)).$$

As is seen Theorem 2.6 in [1], integrating from 0 to t, we can determine the constant C_1 in Proposition 2.1.

We establish the following result on the solvability of the equation (SE).

Theorem 2.1. *Let A and the nonlinear mapping g be given satisfying the assumptions mentioned above. Then, for any $(\Phi^0, \Phi^1) \in H \times L^2(-h, 0; V)$ and $f \in L^2(0, T; V^*)$, $T > 0$, the following nonlinear equation*

$$\begin{cases} x'(t) + Ax(t) = G(t, x) + f(t), & 0 < t \le T, \\ x(0) = \phi^0, \quad x(s) = \phi^1(s), & -h \le s \le 0 \end{cases} \tag{2.6}$$

has a unique solution x belonging to

$$L^2(-h, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H)$$

and satisfying that there exists a constant C_2 such that

$$||x||_{L^2(-h,T;V)\cap W^{1,2}(0,T;V*)} \le C_2(1 + |\phi^0| + ||\phi^1||_{L^2(-h,0;V)} + ||f||_{L^2(0,T;V*)}). \tag{2.7}$$

Proof. Let $y \in L^2(0, T; V)$. Then, we extend it to the interval $(-h, 0)$ by setting $y(s) = \Phi^1(s)$ for $s \in (-h, 0)$, and hence, $G(\cdot, y(\cdot)) \in L^2(0, T; H)$ from Lemma 2.2. Thus, by virtue of Proposition 2.1, we know that the problem

$$\begin{cases} x'(t) + Ax(t) = G(t, y) + f(t), & 0 < t, \\ x(0) = \phi^0, \quad x(s) = \phi^1(s) & -h \le s \le 0 \end{cases} \tag{2.8}$$

has a unique solution $x_y \in L^2(0, T; V) \cap W^{1,2}(0, T; V^*)$ corresponding to y. Let us fix $T_0 > 0$ so that

$$\omega_1^{-1}e^{\omega_2 T_0}(L_1\sqrt{T_0} + L_2K_1T_0^{3/2}/\sqrt{2}) < 1. \tag{2.9}$$

Let x_i, $i = 1, 2$, be the solution of (2.8) corresponding to y_i. Multiplying by $x_1(t) - x_2(t)$, we have that

$$\begin{aligned}&(\dot{x}_1(t) - \dot{x}_2(t), x_1(t) - x_2(t)) + (Ax_1(t) - Ax_2(t), x_1(t) - x_2(t))\\ &= (G(t, y_1) - G(t, y_2), x_1(t) - x_2(t)),\end{aligned}$$

and hence it follows that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|x_1(t) - x_2(t)|^2 + \omega_1||x_1(t) - x_2(t)||^2\\ &\le \omega_2|x_1(t) - x_2(t)|^2 + ||G(t, y_1)) - G(t, y_2)||_*||x_1(t) - x_2(t)||.\end{aligned}$$

From Lemma 2.2 and integrating over $[0,t]$, it follows

$$\begin{aligned}&\frac{1}{2}|x_1(t) - x_2(t)|^2 + \omega_1\int_0^t ||x_1(s) - x_2(s)||^2 ds\\ &\le \frac{1}{2c}\int_0^t ||G(s, y_1) - G(s, y_2)||_*^2 ds\\ &+\frac{c}{2}\int_0^t ||x_1(s) - x_2(s)||^2 ds + \omega_2\int_0^t |x_1(s)ds - x_2(s)ds|^2 ds,\end{aligned}$$

where c is a positive constant satisfying $2\omega_1$ - c >0. Here, we used that

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}, \quad p^{-1} + q^{-1} = 1(1 < p < \infty)$$

for any pair of nonnegative numbers a and b. Thus, from (2.3) it follows that

$$\begin{aligned}
&|x_1(t) - x_2(t)|^2 + (2\omega_1 - c)\int_0^t ||x_1(s)ds - x_2(s)ds||^2 ds \\
&\le c^{-1}(L_1\sqrt{T_0} + L_2K_1T_0^{3/2}/\sqrt{2})^2 \int_0^t ||y_1(s) - y_2(s)||^2 ds \\
&+2\omega_2 \int_0^t |x_1(s) - x_2(s)|^2 ds.
\end{aligned}$$

By using Gronwall's inequality, we get

$$\begin{aligned}
&|x_1(T_0) - x_2(T_0)|^2 + (2\omega_1 - c)\int_0^{T_0} ||x_1(s) - x_2(s)||^2 ds \\
&\le c^{-1}(L_1\sqrt{T_0} + L_2K_1T_0^{3/2}/\sqrt{2})^2 e^{2\omega_2 T_0} \int_0^{T_0} ||y_1(s) - y_2(s)||^2 ds.
\end{aligned}$$

Taking $c = \omega_1$, it holds that

$$\begin{aligned}
||x_1 - x_2||_{L^2(0,T_0;V)} \le \omega_1^{-1} e^{\omega_2 T_0}(&L_1\sqrt{T_0} \\
&+ L_2K_1T_0^{3/2}/\sqrt{2})||y_1 - y_2||_{L^2(0,T_0;V)}.
\end{aligned}$$

Hence, we have proved that $y \mapsto x$ is a strictly contraction from $L^2(0, T_0; V)$ to itself if the condition (2.9) is satisfied. It shows that the equation (2.6) has a unique solution in $[0, T_0]$.

From now on, we derive the norm estimates of solution of the equation (2.6). Let y be the solution of

$$\begin{cases} y'(t) + Ay(t) = f(t), & 0 < t \le T_0, \\ y(0) = \phi^0. \end{cases} \tag{2.10}$$

Then,

$$\frac{d}{dt}(x(t) - y(t)) + (Ax(t) - Ay(t)) = G(t,x),$$

by multiplying by $x(t)$ - $y(t)$ and using the assumption (A1), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|x(t) - y(t)|^2 + \omega_1 ||x(t) - y(t)||^2 \\
&\le \omega_2 |x(t) - y(t)|^2 + ||G(t,x)||_* ||x(t) - y(t)||.
\end{aligned}$$

By integrating over $[0, t]$ and using Gronwall's inequality, we have

$$\begin{aligned}
&||x - y||_{L^2(0,T_0;V)} \le \omega_1^{-1} e^{\omega_2 T_0} ||G(\cdot, x)||_{L^2(0,T_0;V*)} \\
&\le \omega_1^{-1} e^{\omega_2 T_0}\{L_0\sqrt{T_0} + L_2K_0T_0^{3/2}/\sqrt{3} \\
&\quad + (L_1\sqrt{T_0} + L_2K_1T_0^{3/2}/\sqrt{2})(||x||_{L^2(0,T_0;V)} + ||\phi^1||_{L^2(-h,0;V)})\},
\end{aligned}$$

and hence, putting

$$N = \omega_1^{-1} e^{\omega_2 T_0} \text{ and } L = L_1\sqrt{T_0} + L_2K_1T_0^{3/2}/\sqrt{2},$$

it holds

$$\begin{aligned}
&||x||_{L^2(0,T_0;V)} \leq \frac{N}{1-NL}(L_0\sqrt{T_0} + L_2K_0T_0^{3/2}/\sqrt{3}) \\
&+ \frac{1}{1-NL}||y||_{L^2(0,T_0;V)} + \frac{NL}{1-NL}||\phi^1||_{L^2(-h,0;V}) \\
&\leq \frac{N}{1-NL}(L_0\sqrt{T_0} + L_2K_0T_0^{3/2}/\sqrt{3}) \\
&+ \frac{C_1}{1-NL}(|\phi^0| + ||f||_{L^2(0,T_0;V*)}) \\
&+ \frac{NL}{1-NL}||\phi^1||_{L^2(-h,0;V)} \\
&\leq C_2(1 + |\phi^0| + ||\phi^1||_{L^2(-h,0;V)} + ||f||_{L^2(0,T_0;V*)})
\end{aligned} \tag{2.11}$$

for some positive constant C_2. Since the condition (2.9) is independent of initial values, the solution of (2.6) can be extended to the internal [0, nT_0] for natural number n, i.e., for the initial value ($x(nT_0)$, x_nT_O) in the interval [nT_0, $(n + 1)T_0$], as analogous estimate (2.11) holds for the solution in [0, $(n + 1)T_0$].

Theorem 2.2. *If* (Φ^0, Φ^1) ∈ $H \times L^2(-h, 0, V))$ *and* $f \in L^2(0, T; V^*)$, *then* $x \in L^2(-h, T; V))\cap W^{1,2}(0, T; V^*)$, *and the mapping*

$$H \times L^2(-h,0;V) \times L^2(0,T;V^*) \ni (\phi^0, \phi^1, f) \mapsto x \in L^2(-h,T;V)) \cap W^{1,2}(0,T;V^*)$$

is continuous.

Proof. It is easy to show that if (Φ^0, Φ^1) ∈ $H \times L^2(-h, 0; V))$ and $f \in L^2(0, T; V^*)$ for every $T > 0$, then x belongs to $L^2(-h, T; V)\cap W^{1,2}(0, T; V^*)$. Let

$$(\phi_i^0, \phi_i^1, f_i) \in H \times L^2(-h,0;V) \times L^2(0,T_1;V^*)$$

and x_i be the solution of (2.6) with $(\phi_i^0, \phi_i^1, f_i)$ in place of (ϕ^0, ϕ^1, f) for i = 1,2.
Then, in view of Proposition 2.1 and Lemma 2.2, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|x_1(t) - x_2(t)|^2 + \omega_1||x_1(t) - x_2(t)||^2 \\
&\leq \omega_2|x_1(t) - x_2(t)|^2 + ||G(t, x_1) - G(t, x_2)||_*||x_1(t) - x_2(t)|| \\
&+||f_1(t) - f_2(t)||_*||x_1(t) - x_2(t)||
\end{aligned} \tag{2.12}$$

If $\omega_1 - c/2 > 0$, we can choose a constant $c_1 > 0$ so that

$$\omega_1 - c/2 - c_1/2 > 0$$

and

$$\begin{aligned}
||f_1(t) - f_2(t))||_*||x_1(t) - x_2(t)|| \leq &\frac{1}{2c_1}||f_1(t) - f_2(t)||_*^2 \\
&+ \frac{c_1}{2}||x_1(t) - x_2(t)||^2.
\end{aligned}$$

Let $T_1 < T$ be such that

$$2\omega_1 - c - c_1 - c^{-1}e^{2\omega_2 T_1}(L_1\sqrt{T_1} + L_2K_1T_1^{3/2}/\sqrt{2})^2 > 0.$$

Integrating on (2.12) over $[0, T_1]$ and as is seen in the first part of proof, it follows

$$\begin{aligned}
&(2\omega_1 - c - c_1)||x_1 - x_2||^2_{L^2(0,T_1;V)} \le e^{2\omega_2 T_1}\{|\varphi_1^0 - \varphi_2^0|^2 \\
&+ \frac{1}{c}||G(t,x_1) - G(t,x_2)||^2_{L^2(0,T_1;V*)} + \frac{1}{c_1}||f_1 - f_2||^2_{L^2(0,T_1;V*)}\} \\
&\le e^{2\omega_2 T_1}\{|\phi_1^0 - \phi_2^0|^2 \\
&+ \frac{1}{c}(L_1\sqrt{T}_1 + L_2K_1T_1^{3/2}/\sqrt{2})^2||x_1 - x_2||^2_{L^2(-h,T_1;V)} \\
&+ \frac{1}{c_1}||f_1 - f_2||^2_{L^2(0,T_1;V*)}\}.
\end{aligned}$$

Putting that

$$N_1 = 2\omega_1 - c - c_1 - c^{-1}e^{2\omega_2 T_1}(L_1\sqrt{T}_1 + L_2K_1T_1^{3/2}/\sqrt{2})^2$$

we have

$$\begin{aligned}
||x_1 - x_2||_{L^2(0,T_1;V)} &\le \frac{e^{\omega_2 T_1}}{N_1^{1/2}}(|\phi_1^0 - \phi_2^0| + \frac{1}{c_1}||f_1 - f_2||_{L^2(0,T_1;V*)}) \\
&+ \frac{c^{-1/2}e^{\omega_2 T_1}(L_1\sqrt{T_1} + L_2K_1T_1^{3/2}/\sqrt{2})}{N_1^{1/2}}||\phi_1^1 - \phi_2^1||_{L^2(-h,0;V)}.
\end{aligned} \tag{2.13}$$

Suppose that

$$(\phi_n^0, \phi_n^1, f_n) \to (\phi^0, \phi^1, f) \text{ in } H \times L^2(-h, 0; V)) \times L^2(0, T; V^*),$$

and let x_n and x be the solution (2.6) with $(\phi_n^0, \phi_n^1, f_n)$ and (ϕ^0, ϕ^1, f) respectively.

By virtue of (2.13) with T being replaced by T_1, we see that

$$x_n \to x \quad \text{in} \quad L^2(-h, T_1; V)) \cap W^{1,2}(0, T_1; V^*) \subset C([0, T_1]; H).$$

This implies that $(x_n(T_1), (x_n)_{T_1}) \to (x(T_1), x_{T_1})$ in $H \times L^2$ $(-h, 0; V)$. Hence, the same argument shows that

$$x_n \to x \quad \text{in} \quad L^2(T_1, \min\{2T_1, T\}; V) \cap W^{1,2}(T_1, \min\{2T_1, T\}; V^*).$$

Repeating this process, we conclude that

$$x_n \to x \quad \text{in} \quad L^2(-h, T; V) \cap W^{1,2}(0, T; V^*).$$

Remark 2.2. *For $x \in L^2(0, T; V)$, we set*

$$G(t, x) = \int_0^t k(t-s)g(s, x(s))ds$$

where k belongs to $L^2(0, T)$ and $g : [0, T] \times V \to H$ be a nonlinear mapping satisfying

$$|g(t, x) - g(t, y)| \le L||x - y||$$

for a positive constant L. Let $x \in L^2(0, T; V)$, $T > 0$. Then, $G(\cdot, x) \in L^2(0, T; H)$ and

$$||G(\cdot, x)||_{L^2(0,T;H)} \le L||k||_{L^2(0,T)}\sqrt{T}||x||_{L^2(0,T;V)}.$$

Moreover, if $x_1, x_2 \in L^2(0, T; V)$, then

$$||G(\cdot, x_1) - G(\cdot, x_2)||_{L^2(0,T;H)} \le L||k||\sqrt{T}||x_1 - x_2||_{L^2(0,T;V)}.$$

Then, with the condition that

$$\omega_1^{-1} e^{\omega_2 T_0} L||k||\sqrt{T_0} < 1$$

in place of the condition (2.9), we can obtain the results of Theorem 2.1.

3 Approximate controllability

In what follows we assume that the embedding $V \subset H$ is compact, and A is a continuous operator from V to V^* satisfying (A1) and (A2). For $h \in L^2(0, T; H)$ and let x_h be the solution of the following equation with $B = I$:

$$\begin{cases} x'(t) + Ax(t) = G(t, x) + h(t), & 0 < t, \\ x(0) = 0 \quad x(s) = 0 & -h \le s \le 0, \end{cases} \tag{3.1}$$

where

$$G(t, x) = g(t, x_t, \int_0^t k(t, s, x_s)ds).$$

We define the solution mapping S from $L^2(0, T; V^*)$ to $L^2(0, T; V)$ by

$$(Sh)(t) = x_h(t), \; h \in L^2(0, T; V^*). \tag{3.2}$$

Let $\mathcal{A}$ and $\mathcal{G}$ be the Nemitsky operators corresponding to the maps A and G, which are defined by $\mathcal{A}(x)(\cdot) = Ax(\cdot)$ and $\mathcal{G}(h)(\cdot) = G(\cdot, x_h)$, respectively. Then, since the solution x belongs to $L^2(-h, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H)$, it is represented by

$$x_h(t) = \int_0^t ((I + \mathcal{G} - \mathcal{A}S)h)(s)ds, \tag{3.3}$$

and with aid of Lemma 2.2 and Proposition 2.1

$$\begin{aligned} &||Sh||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} = ||x_h|| \le C_1||G(\cdot, x_h) + h||_{L^2(0,T;V*)} \\ &\le C_1\{L_0\sqrt{T} + L_2K_0T^{3/2}/\sqrt{3} + (L_1\sqrt{T} + L_2K_1T^{3/2}/\sqrt{2})||x||_{L^2(0,T;V)} \\ &+||h||_{L^2(0,T;V*)}\} \\ &\le C_1\{L_0\sqrt{T} + L_2K_0T^{3/2}/\sqrt{3} \\ &+(L_1\sqrt{T} + L_2K_1T^{3/2}/\sqrt{2})(1 + ||h||_{L^2(0,T;V*)}) + ||h||_{L^2(0,T;V*)}\}. \end{aligned} \tag{3.4}$$

Hence, if h is bounded in $L^2(0, T; V^*)$, then so is x_h in $L^2(0, T; V) \cap W^{1,2}(0, T; V^*)$. Since V is compactly embedded in H by assumption, the embedding $L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset L^2(0, T; H)$ is compact in view of Theorem 2 of Aubin [14]. Hence, the mapping $h \mapsto Sh = x_h$ is compact from $L^2(0, T; V^*)$ to $L^2(0, T; H)$. Therefore, $\mathcal{G}$ is a compact mapping from $L^2(0, T; V^*)$ to $L^2(0, T; H)$ and so is $\mathcal{A}S$ from $L^2(0, T; V^*)$ to itself. The solution of (SE) is denoted by $x(T; g, u)$ associated with the nonlinear term g and control u at time T.

Definition 3.1. *The system (SE) is said to be approximately controllable at time T if* $Cl\{x(T; g, u): u \in L^2(0, T; U)\} = V^*$ *where Cl denotes the closure in* V^*.

We assume

(**T**) $1 - \omega_1^{-1}\omega_3 e^{\omega_2 T} > 0$

(**B**) $Cl\{y : y(t) = (Bu)(t)$, a.e. $u \in L^2(0, T; U)\} = L^2(0, T; U)\}$. Here Cl is the closure in $L^2(0, T; H)$.

Theorem 3.1. *Let the assumptions (T) and (B) be satisfied. Then,*

$$Cl\{(I-\mathcal{A}S)h : h \in L^2(0,T;V^*)\} = L^2(0,T;V^*). \tag{3.5}$$

Therefore, the following nonlinear differential control system

$$\begin{cases} \dfrac{dx(t)}{dt} + Ax(t) = (Bu)(t), \quad 0 < t \leq T, \\ x(0) = x_0 \end{cases} \tag{3.6}$$

is approximately controllable at time T.

Proof. Let $z \in L^2(0, T; V^*)$ and r be a constant such that

$$z \in U_r = \{x \in L^2(0,T;V^*) : ||x||_{L^2(0,T;V^*)} < r\}.$$

Take a constant $d > 0$ such that

$$(r + \omega_3 + N_2|x_0|)(1-N_2)^{-1} < d, \tag{3.7}$$

where

$$N_2 = \omega_1^{-1}\omega_3 e^{\omega_2 T}.$$

Taking scalar product on both sides of (3.1) with $G = 0$ by $x(t)$

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|x(t)|^2 + \omega_1||x(t)||^2 &\leq \omega_2|x(t)|^2 + ||h(t)||_*||x(t)|| \\ &\leq \omega_2|x(t)|^2 + \frac{1}{2c}||h(t)||_*^2 + \frac{c}{2}||x(t)||^2 \end{aligned}$$

where c is a positive constant satisfying $2\omega_1 - c > 0$. Integrating on $[0, t]$, we get

$$\begin{aligned} \frac{1}{2}|x(t)|^2 + \omega_1\int_0^t ||x(s)||^2 ds &\leq \frac{1}{2}|x_0|^2 + \frac{1}{2c}\int_0^t ||h(s)||_*^2 ds \\ &\quad + \frac{c}{2}\int_0^t ||x(s)||^2 ds + \omega_2\int_0^t |x(s)|^2 ds, \end{aligned}$$

and hence,

$$\begin{aligned} |x(t)|^2 + (2\omega_1 - c)\int_0^t ||x(s)||^2 ds \leq{} & |x_0|^2 + \frac{1}{c}\int_0^t ||h(s)||_*^2 ds \\ & + 2\omega_2\int_0^t |x(s)|^2 ds. \end{aligned}$$

By using Gronwall's inequality, it follows that

$$|x(T)|^2 + (2\omega_1 - c)\int_0^T ||x(s)||^2 ds \leq e^{2\omega_2 T}(|x_0|^2 + \frac{1}{c}\int_0^T ||h(s)||_*^2 ds),$$

that is,

$$\begin{aligned} ||Sh||_{L^2(0,T;V)} &= ||x||_{L^2(0,T;V)} \\ &\leq e^{\omega_2 T}(2\omega_1 - c)^{-1/2}(|x_0| + c^{-1/2}||h||_{L^2(0,T;V^*)}). \end{aligned} \tag{3.8}$$

Let us consider the equation

$$z = (I - \mathcal{A}S)w. \tag{3.9}$$

Let w be the solution of (3.9). Then $z \in U_d$ and taking $c = \omega_1$, from (3.7), (3.8)

$$\begin{aligned}||w||_{L^2(0,T;V^*)} &\leq ||z||_{L^2(0,T;V^*)} + ||\mathcal{AS}w||_{L^2(0,T;V^*)}\\ &\leq r + \omega_3(||Sw||_{L^2(0,T;V^*)} + 1)\\ &\leq r + \omega_3\{\omega_1^{-1/2}e^{\omega_2 T}(|x_0| + \omega_1^{-1/2}||w||) + 1\},\end{aligned}$$

and hence

$$||w|| \leq (r + \omega_3 + N_2|x_0|)(1 - N_2)^{-1} < d$$

it follows that $w \notin \partial U_d$ where ∂U_d stands for the boundary of U_d. Thus, the homotopy property of topological degree theory there exists $w \in L^2(0, T; V^*)$ such that the equation (3.9) holds. Based on the assumption (B), there exists a sequence $\{u_n\} \in L^2(0, T; U)$ such that $Bu_n \mapsto w$ in $L^2(0, T; V^*)$. Then, by the last paragraph of Theorem 2.1, we have that $x(\cdot; g, u_n) \mapsto x_w$ in $L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H)$. Hence, we have proved (3.5). Let $y \in V^*$. Then, there exists an element $u \in L^2(0, T; U)$ such that

$$||\frac{y}{T} - (I - \mathcal{A}S)Bu||_{L^2(0,T;V^*)} < \frac{\varepsilon}{\sqrt{T}}.$$

Thus

$$\begin{aligned}||y - x(T)||_* &= ||y - \int_0^T ((I - \mathcal{A}S)Bu)(s)ds||_*\\ &\leq \int_0^T ||\frac{y}{T} - ((I - \mathcal{A}S)Bu)(s)||_* ds\\ &\leq \sqrt{T}||\frac{y}{T} - (I - \mathcal{A}S)Bu||_{L^2(0,T;V*)} < \varepsilon.\end{aligned}$$

Therefore, the system (3.6) is approximately controllable at time T.

In order to investigate the controllability of the nonlinear control system, we need to impose the following condition.

(F) g is uniformly bounded: there exists a constant M_g such that

$$|g(t, x, y)| \leq M_g,$$

for all $x, y \in V$.

By (F) it holds that

$$||G(\cdot, x)||_{L^2(0,T;H)} \leq M_g\sqrt{T},$$

and for every $h \in L^2(0, T; V^*)$

$$||\mathcal{G}(h)||_{L^2(0,T;H)} \leq M_g\sqrt{T} \tag{3.10}$$

Theorem 3.2. *Let the assumptions (T), (B), and (F) be satisfied. Then, we have*

$$Cl\{(\mathcal{G} + I - \mathcal{A}S)h : h \in L^2(0, T; V^*)\} = L^2(0, T; V^*). \tag{3.11}$$

Thus, the system (SE) is approximately controllable at time T.

Proof. Let U_r be the ball with radius r in $L^2(0, T; V^*)$ and $z \in U_r$. To prove (3.11), we will also use the degree theory for the equation

$$z = \lambda(\mathcal{G} - \mathcal{A}S)w + w, \quad 0 \leq \lambda \leq 1 \tag{3.12}$$

in open ball U_d where the constant d satisfies

$$(r + \omega_3 + N_2|x_0| + M_g\sqrt{T})(1 - N_2)^{-1} < d \tag{3.13}$$

where the constant N_2 is in Theorem 3.1. If w is the solution of (3.12), then $z \in U_d$ and from Lemma 2.1

$$\begin{aligned}||w||_{L^2(0,T;V*)} &\le ||z|| + ||\mathcal{A}Sw|| + ||\mathcal{G}w|| \\ &\le r + \omega_3(||Sw|| + 1) + M_g\sqrt{T}) \\ &\le r + \omega_3\{\omega_1^{-1/2}e^{\omega_2 T}(|x_0| + \omega_1^{-1/2}||w||) + 1\} + M_g\sqrt{T},\end{aligned}$$

and hence

$$||w|| \le (r + \omega_3 + N_2|x_0| + M_g\sqrt{T})(1 - N_2)^{-1} < d$$

it follows that $w \notin \partial U_d$. Hence, there exists $w \in L^2(0, T; V^*)$ such that the equation (3.12) holds. Using the similar way to the last part of Theorem 3.1 and the assumption (B), there exists a sequence $\{un\} \in L^2(0, T; U)$ such that $Bu_n \mapsto w$ in $L^2(0, T; V^*)$ and $x(\cdot, g, u_n) \mapsto x_w$ in $L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H)$. Thus, we have proved (3.11), and the system (1.1) is approximately controllable at time T.

4 Example

Let $-A$ be an operator associated with a bounded sesquilinear form $a(u, v)$ defined in $V \times V$ and satisfying Gårding inequality

$$\operatorname{Re} a(u, v) \ge c_0||u||^2 - c_1|u|^2, \quad c_0 > 0, \quad c_1 \ge 0$$

for any $u \in V$. It is known that A generates an analytic semigroup in both H and V^*. By virtue of the Riesz-Schauder theorem, if the embedding $V \subset H$ is compact, then the operator A has discrete spectrum:

$$\sigma(A) = \{\mu_n : n = 1, 2, \dots\}$$

which has no point of accumulation except possibly when $\mu = \infty$. Let μ_n be a pole of the resolvent of A of order k_n and P_n the spectral projection associated with μ_n

$$P_n = \frac{1}{2\pi i}\int_{\Gamma_n}(\mu - A)^{-1}d\mu,$$

where Γ_n is a small circle centered at μ_n such that it surrounds no point of $\sigma(A)$ except μ_n. Then, the generalized eigenspace corresponding to μ_n is given by

$$H_n = P_nH = \{P_nu : u \in H\},$$

and we have that from $P_n^2 = P_n$ and $H_n \subset V$; it follows that

$$P_nV = \{P_nu : u \in V\} = H_n.$$

Definition 4.1. *The system of the generalized eigenspaces of A is complete in H if* Cl $\{\text{span}\{H_n : n = 1, 2,\dots\}\} = H$ *where* Cl *denotes the closure in H.*

We need the following hypotheses:

(B1) The system of the generalized eigenspaces of A is complete.

(B2) There exists a constant d >0 such that

$$||\upsilon|| \leq d||B\upsilon||, \quad \upsilon \in L^2(0,T;U).$$

We can see many examples which satisfy (B2)(cf. [8,11]).

Consider about the intercept controller B define d by

$$(Bu)(t) = \sum_{n=1}^{\infty} u_n(t), \tag{4.1}$$

where

$$u_n = \begin{cases} 0, & 0 \leq t \leq \dfrac{T}{n} \\ P_n u(t), & \dfrac{T}{n} < t \leq T. \end{cases}$$

Hence, we see that $u_1(t) \equiv 0$ and $u_n(t) \in \text{Im } P_n$.

First of all, for the meaning of the condition (B) in section 3, we need to show the existence of controller satisfying $\text{Cl}\{Bu : u \in L^2(0, T; U)\} \neq L^2(0, T; H)$. In fact, by completion of the generalized eigenspaces of A, we may write that $f(t) = \sum_{n=1}^{\infty} P_n f(t)$ for $\in L^2(0, T; H)$. Let us choose $f \in L^2(0, T; H)$ satisfying

$$\int_0^T ||P_1 f(t)||^2 dt > 0.$$

Then, since

$$\int_0^T ||f(t) - Bu(t)||^2 dt = \int_0^T \sum_{n=1}^{\infty} ||P_n(f(t) - Bu(t))||^2 dt$$
$$\geq \int_0^T ||P_1(f(t) - Bu(t))||^2 dt = \int_0^T ||P_1 f(t)||^2 dt > 0,$$

the statement mentioned above is reasonable.

Let $f \in L^2(0, T; H)$ and $\alpha = T/(T - T/n)$. Then we know

$$f(\cdot) \equiv \alpha K_{[T,T/n]} f(\alpha(\,\cdot\, - T/n)) \quad \text{in} \quad L^2(0,T;H),$$

where $K_{[T,T/n]}$ is the characteristic of $[T,T/n]$. Define

$$w(s) = \sum_{n=1}^{\infty} w_n(s), \quad w_n(s) = \alpha K_{[T,T/n]} B^{-1} P_n f(\alpha(s - T/n)).$$

Thus $(Bw)(t) = \sum_{n=1}^{\infty} P_n f(s)$, a.e. Since the system of the generalized eigenspaces of A is complete, it holds that for every $f \in L^2(0, T; H)$ and $\in > 0$

$$||f(\cdot) - \sum_{n=1}^{\infty} P_n f(\cdot)||_{L^2(0,T;H)} = ||f(\cdot) - Bw||_{L^2(0,T;H)} < \varepsilon.$$

Thus, the intercept controller B define d by (4.1) satisfies the condition (B).

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JMJ carried out the main proof of this manuscript, JRK drafted the manuscript and corrected the main theorems, EYJ conceived of the study, and participated in its design and coordination.

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The authors declare that they have no competing interests.

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